Extremely Fast Numerical Integration of Ocean Surface Wave Dynamics: Building Blocks for a Higher Order Method

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LONG-TERM GOALS

1. Application of ***multidimensional Fourier analysis for the extremely fast numerical integration of the partial differential equations of surface water waves*** is the long-term goal of this work. The approach is a generalization of linear Fourier analysis and therefore is a ***hyperfast extension*** of the ***fast Fourier transform*** *(FFT)* to the ***nonlinear Euler water wave equations***.

2. The present report discusses progress made in both the shallow and deep-water environments, anticipating the marriage of the two in the coming year for a complete, higher order, hyperfast integration to Euler. An important application of the approach is possibility of the ***on-board computation of wave properties as derived from shipboard direct wave and/or radar measurements of the sea surface***, leading to ***prediction of sea state conditions***, including the presence of ***rogue waves*** in ***real time.***

OBJECTIVES

1. The objective of the present research program is the development of ***fast numerical multidimensional Fourier*** techniques applied to a wide range of wave modeling and analysis problems.

2. Important progress made in the past year has been the ***development of three new algorithms for multidimensional Fourier analysis***. These algorithms are the key to future ***hyperfast applications of the method***.

APPROACH

We first consider the shallow water equation known as the ***Korteweg-deVries*** (KdV) equation):

$$\eta_t + c_o \eta_x + \alpha \eta \eta_x + \beta \eta_{xxx} = 0 \tag{1}$$

where $c_o = \sqrt{gh}$, $\alpha = 3c_o / 2h$ and $\beta = c_o h^2 / 6$. The KdV equation has the generalized Fourier solution (for periodic and/or quasi-periodic boundary conditions) that is found by the ***inverse scattering transform*** (see [Osborne, 2002] and cited references):

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$$\eta(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\theta(x,t\,|\tilde{\mathbf{B}},\mathbf{k},\boldsymbol{\omega},\boldsymbol{\phi}) \tag{2}$$

The ***generalized Fourier series***, $\theta(x,t\,|\tilde{\mathbf{B}},\mathbf{k},\boldsymbol{\omega},\boldsymbol{\phi})$, is given by the expression

$$\theta(x,t\,|\,\mathbf{B},\boldsymbol{\phi}) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_N=-\infty}^{\infty} e^{i\sum_{n=1}^{N} m_n X_n - \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} m_m m_n B_{mn}} \tag{3}$$

where $X_n = k_n x - \omega_n t + \phi_n$. The function $\theta(x,t\,|\tilde{\mathbf{B}},\boldsymbol{\phi})$ is also called a ***Riemann theta function*** or ***multidimensional Fourier series***. Here **B** is the Riemann matrix (the "spectrum" of the solution), the vector **k** constitutes the usual wave numbers, the vector $\boldsymbol{\omega}$ contains the frequencies and the vector $\boldsymbol{\phi}$ forms the phases. The ***inverse problem*** associated with (2), (3) allows one to determine the Riemann matrix, wave numbers, frequencies and phases appropriate for solving the *Cauchy problem* for KdV: Given the spatial variation of the solutions at $t = 0$, $\eta(x,0)$, compute the solution for all time, $\eta(x,t)$. This is a necessary step for the numerical simulations presented heein. The ***solitons, Stokes waves and sine waves lie on the diagonal of the Riemann matrix***; the ***off-diagonal terms contain the nonlinear interactions***.

Why is the above approach useful for hyperfast numerical simulations? Because the Riemann theta function can be programmed as a ***fast theta function transform*** (FTFT), just as the Fourier transform can be programmed as a ***fast Fourier transform*** (FFT). Therefore the numerical integration of KdV (1) can be evaluated at specific time points, necessary only for graphical purposes or for extracting useful properties of the sea surface. This contrasts to the FFT that must be evaluated at very small time steps for the accurate integration of a nonlinear partial differential equation.

We now consider the case for ***directional spreading***. This occurs in the shallow water equation known as the **Kadomtsev-Petviashvili** (KP) equation):

$$(\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx})_x + \gamma\eta_{yy} = 0 \tag{4}$$

where $\gamma = c_o / 2$. The KdV equation has the generalized Fourier solution (for periodic boundary conditions) that is found by the ***inverse scattering transform*** (see [Osborne, 2002] and cited references):

$$\eta(x,y,t) = 2\frac{\partial^2}{\partial x^2}\ln\theta(x,y,t\,|\tilde{\mathbf{B}},\mathbf{k},\boldsymbol{\omega},\boldsymbol{\phi}) \tag{5}$$

The ***generalized Fourier series***, $\theta(x,t\,|\tilde{\mathbf{B}},\mathbf{k},\boldsymbol{\omega},\boldsymbol{\phi})$, is given by (3) above, where now $X_n = k_n x + l_n y - \omega_n t + \phi_n$. Once again the function $\theta(x,t\,|\tilde{\mathbf{B}},\boldsymbol{\phi})$ is also called a ***Riemann theta function*** or multidimensional Fourier series; **B** is the Riemann matrix, the vector **k** constitutes the wave numbers in the *x* direction and the wave number vector **l** constitutes the *y*-direction wave numbers, the vector $\boldsymbol{\omega}$ contains the frequencies and the vector $\boldsymbol{\phi}$ contains the phases. The *Cauchy problem* for KP

is defined by: Given the spatial variation of the solutions at $t=0$, $\eta(x,y,0)$, compute the solution for all time, $\eta(x,y,t)$. The ***solitons, Stokes waves and sine waves lie on the diagonal of the Riemann matrix*** and are spread in direction according to the wave numbers (**k**, **l**). As before the ***off-diagonal terms contain the nonlinear interactions***. The sea surface elevation corresponds to cnoidal wave basis functions distributed over a directional spectrum, plus nonlinear interactions among the cnoidal waves.

The unidirectional, deep-water case is that for the ***nonlinear Schroedinger equation***:

$$i(\psi_t + C_g\psi_x) + \mu\psi_{xx} + \nu|\psi|^2\,\psi = 0 \tag{6}$$

where the coefficients are computed in the usual way in terms of the carrier wave number and frequency. The spectral solution to this equation is given by

$$\psi(x,t) = a\,\frac{\theta(x,t\,|\,\mathbf{B},\delta^-)}{\theta(x,t\,|\,\mathbf{B},\delta^+)}\,e^{2ia^2 t} \tag{7}$$

Thus the solution is the ratio of two multi-dimensional Fourier series with different phases, $\delta^{\pm}$.

For directionally spread waves in deep-water the nonlinear Schroedinger equation has the form:

$$i(\psi_t + C_g\psi_x) + \mu\psi_{xx} - 2\mu\psi_{yy} + \nu|\psi|^2\,\psi = 0 \tag{8}$$

The spectral solution to this equation, for small directional spreading, is approximated by

$$\psi(x,y,t) \cong a\,\frac{\theta(x,y,t\,|\,\mathbf{B},\delta^-)}{\theta(x,y,t\,|\,\mathbf{B},\delta^+)}\,e^{2ia^2 t} \tag{9}$$

A more complex form is necessary to include large directionally spread waves.

An important result, fundamental in the present research program, is that the ***above four relatively simple cases form building blocks*** for the solution to more complex equations such as the ***Boussinesq equations*** and the full ***Euler equations***. Another important aspect, to be addressed in the coming year, is that ***wind and dissipation forces*** can be easily added to the above equations and to the Euler equations.

WORK COMPLETED

The work completed during the past year includes the coding of three new multidimensional Fourier transform algorithms which are being tested and used to develop ***hyperfast higher order algorithms***.

RESULTS

Here are the codes that have been developed during the past year.

(1) A generalized code for the ***discrete Riemann theta function*** (a multi-dimensional Fourier series) that is designed to work for 1, 2 or 3 dimensions. The discrete nature of the algorithm means that it automatically satisfies ***periodic boundary conditions*** (although other types of boundary conditions will be addressed in the future). This code solves ***nonlinear wave dynamics***, just as the FFT solves linear wave motion.

(2) A new code has been developed which allows for the ***direct computation of the multi-dimensional Fourier spectrum*** (Riemann matrix and phases) without algebraic geometry, which is essential for specification of the initial conditions in numerical modeling efforts. This is viewed as a big step for future development as it does not require complex mathematics and serves as a necessary preprocessor for numerical simulations.

(3) Codes for the shallow water ***Korteweg-deVries, Kadomtsev-Petviashvili and nonlinear Schroedinger equations*** and higher order corrections have been developed as prerequisites to coding the ***Boussinesq*** and ***Euler equations***.

The mathematical formulation and numerical approaches discussed herein are ***phase resolving*** because they contain the phase of the spectral components. This is also true of the more traditional numerical approaches that use the FFT, such as the ***leap-frog method*** and the ***higher order methods***. ***There are essentially three reasons why our methods are faster, by 3 or 4 orders of magnitude, than traditional methods: (1) a fast multidimensional Fourier transform algorithm, (2) a recursion operator formulation and (3) computation of the nonlinear Fourier coefficients, in the solution of the KdV equation for relatively large values of time interval,*** Δt**.**

In Figs. 1 and 2 we show the results of a shallow water simulation using the KP equation in the form of the multi-dimensional Fourier series. This is a case for a fully directional sea state with 50 directionally spread components, corresponding to a 50 by 50 Riemann matrix. The algorithm computes a 100 by 100 grid in the x, y domain. A full simulation, corresponding to about 1 hour using a standard FFT leap-frog code, takes about 2 sec using the present algorithm with multidimensional Fourier series.

Another example, for deep water, is shown in Figs. 3, 4. In this example a simple directional spectrum, corresponding to two unstable rogue wave modes, is shown. The full simulation is about 800 times faster than the standard FFT spectral code.

IMPACT/APPLICATION

The impact of this research will occur in general for the nonlinear Fourier analysis of shallow and deep-water wave trains. Specific results will provide for a deeper understanding of nonlinear wave dynamics.

TRANSITIONS

Transitions expected are related to the use of the codes as guidance to ships and unmanned, unteathered vehicles as the kind of environment in which one resides and for the real time sampling of the environment, including the acoustic environment.

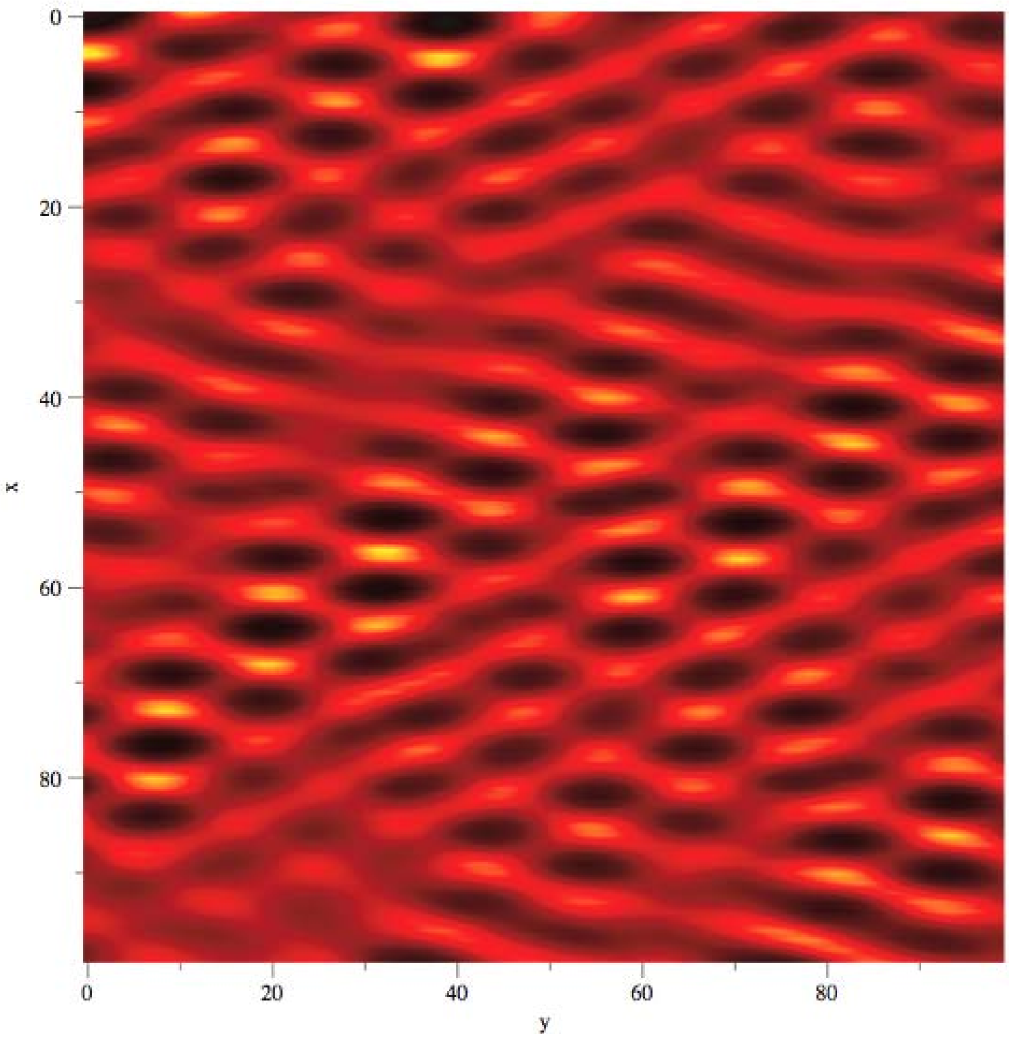

Fig. 1 Image of the sea surface as seen from above for the simulation of a shallow water simple directional spectrum. These results are constructed using the spectral formulation of the Kadomtsev-Petviashvili equation, a standard equation for nonlinear, shallow water wave dynamics that is a cousin of the Boussinesq equation.

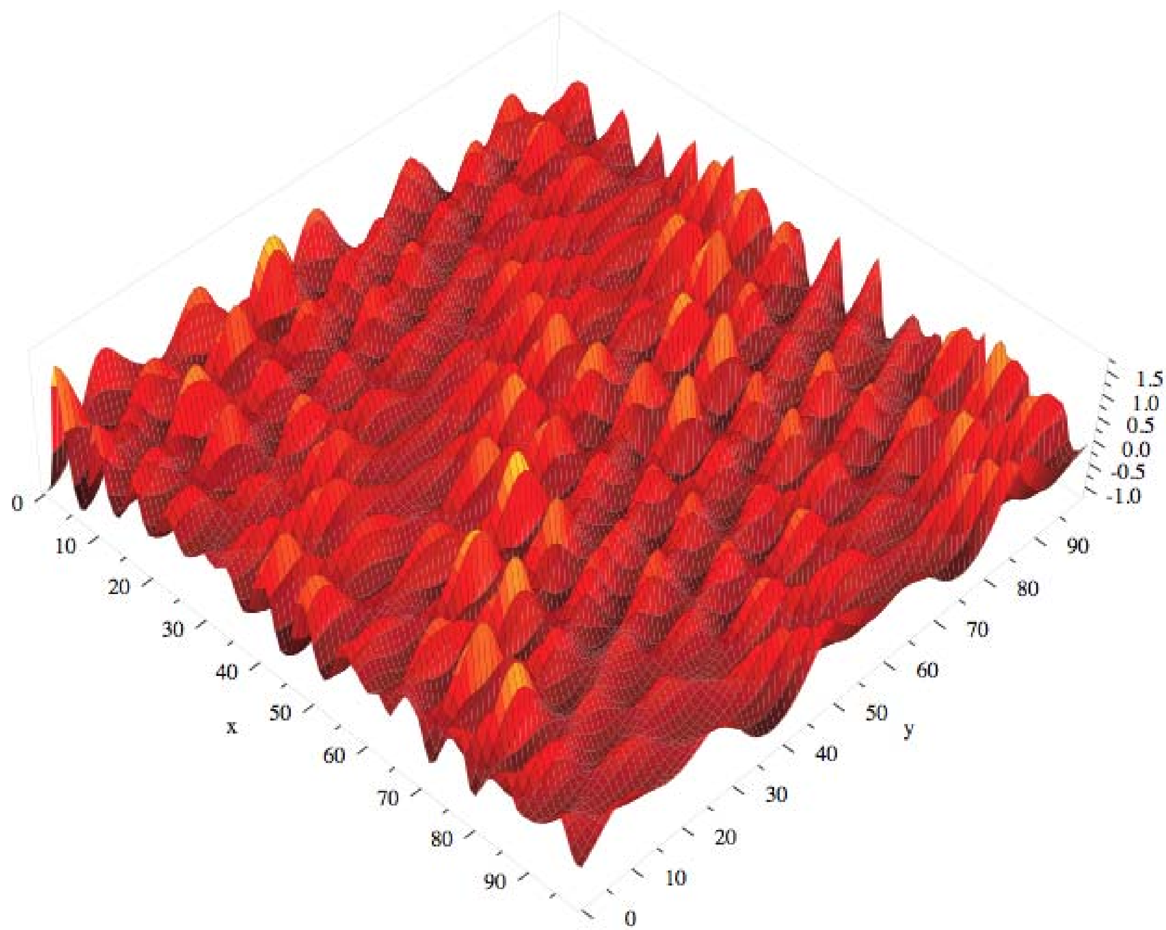

Fig. 2 The surface elevation field of the simulation discussed in Fig. 1 and in the text.

RELATED PROJECTS

An intimate relationship between our results and other projects exists because the sea surface provides a major forcing input to many kinds of offshore activities, including the dynamics of floating and drilling vessels, barges, risers and tethered vehicles. The present work leads to a nonlinear representation of the sea surface forcing and vessel response for shallow water waves.

REFERENCES

A. R. Osborne, Nonlinear Ocean Waves and the Inverse Scattering Transform, in *Scattering*, edited by R. Pike and P. Sabatier, Academic Press, 2001.

PUBLICATIONS

A. R. Osborne, M. Onorato, Modelling Nonlinear, Directionally Spread, Shallow Water Waves Using the Inverse Scattering Transform, submitted, 2006.

A. R. Osborne, M. Onorato, M. Serio and M. Petti, Harmonic generation in shallow water surface waves: Perspective in terms of the inverse scattering transform, submitted, 2005.

A. R. Osborne and M. Onorato, Numerical Computation of Algebraic Geometric Loop Integrals for the Korteweg-deVries Equation, submitted, 2006.

A. R. Osborne and M. Onorato, Infinite Order Closure for the Kinetic Dynamics of the Korteweg-deVries Equation, submitted, 2006.

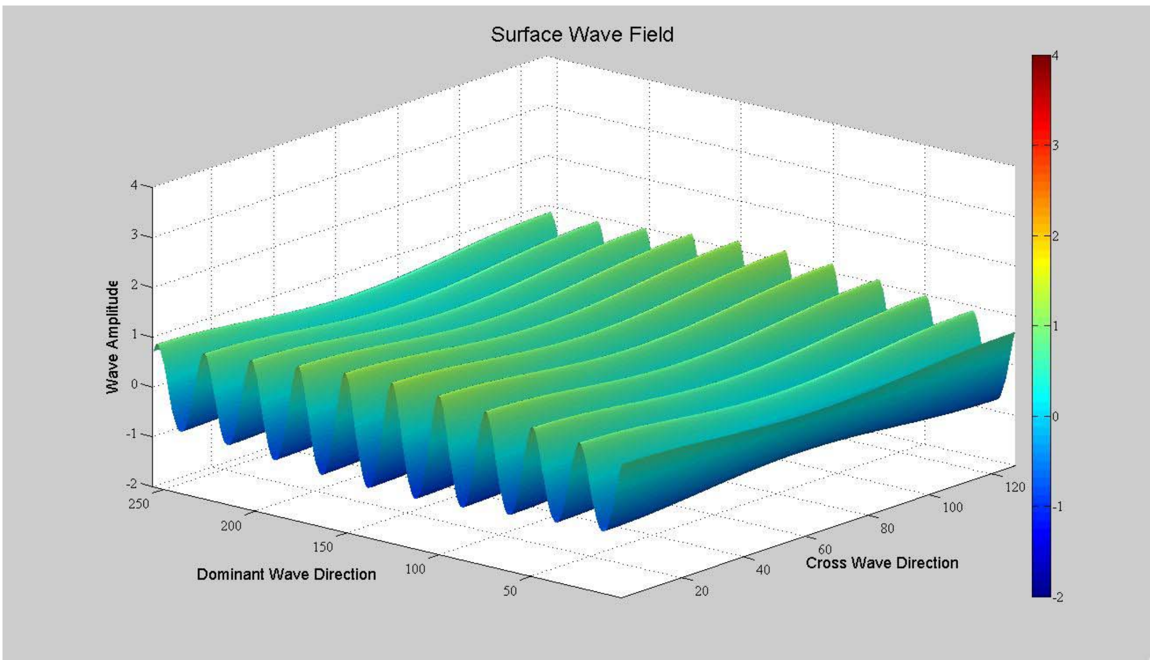

Fig. 3 Initial wave train for deep water using multi-dimensional Fourier series. The wave train is assumed to be a simple sine wave (the "carrier") which is modulated both in the direction of propagation (dominant direction) and in the direction normal to the propagation.

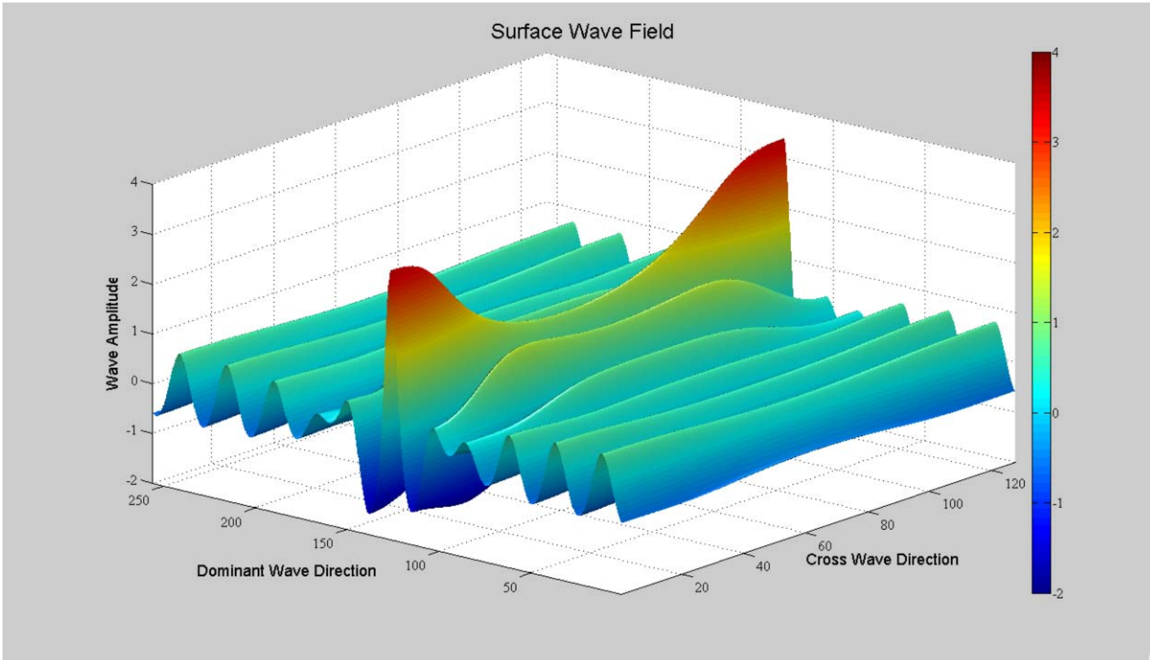

Fig. 4 The wave train in Fig. 3 has grown into a substantial rogue wave that is 3.5 times higher than the amplitude of the initial carrier wave.

PATENTS

None.

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None.